A Theoretical Investigation of Natural Convection Effects in Forced Horizontal Flows

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It is shown that viscosity variations and temperature differences between the upper and lower walls can significantly increase or decrease the tendency toward instability. It was found that low Prandtl number fluids are most sensitive to gravitational field effects because they are characterized by a thicker thermal boundary layer.

Numerous studies relating to buoyancy effects in vertical external and bounded flows have been reported. Ostrach (*14*), Hanratty et al, (*7, 8, 16*), and Maslen (*9*) have considered various aspects of the bounded flow problem and quote numerous other articles. Acrivos (*1*) investigated vertical external flows and presented numerical results for a vertical flat plate which were obtained with the Karman-Pohlhausen integral method. Sparrow et al. (*19*) have obtained similar solutions for vertical boundary-layer flows.

Until the review of this paper by E. R. G. Eckert and E. M. Sparrow was received, the authors were unaware of any previous theoretical work on buoyancy effects in horizontal flows. These reviewers indicated that Mori (*12*) and Morton (*13*) have investigated some of the heat transfer aspects of the problem. Mori's approximate analysis for boundary-layer flow over a constant temperature flat plate employs a series expansion of the stream function; Sparrow (*18*), in a paper to be published, has independently considered the same problem by means of essentially the same method as Mori. Morton obtained an approximate solution for constant property flow in a horizontal pipe with power series expansions in the Rayleigh number. However his results are incomplete since they imply that the secondary flow effect vanishes as $N_{Re} \rightarrow 0$. This occurs because Morton assumed that the axial pressure gradient is constant.

Most theoretical work has assumed that temperature variations in horizontal flows are manifested only in viscosity and inertial density variations in the x-direction momentum equation. With these assumptions it has usually been found for subsonic flows that viscosity variations are the principal cause of instability (*11, 17*). For heated gas flows inflections in the velocity profiles due to viscosity effects are predicted, and in accordance with the Tollmien-Schlichting theory this is a necessary and sufficient condition for instability. In liquids the effect is obviously reversed insofar as viscosity variations are concerned. It will be shown here that for bounded flow, at low N_{Re}, body forces markedly influence the velocity field and to a lesser extent the temperature field; in external or boundary-layer flows, with relatively low free stream velocities, body forces may well play the primary role in determining the flow stability.

Fully developed horizontal flow between infinite parallel plates with a linear axial temperature distribution is considered in some detail. The influence of radial viscosity variations, viscous dissipation, uniform energy generation, and unequal wall temperatures on velocity and temperature fields has been determined, and some of the numerical results are reported.

By means of Prandtl's boundary-layer theory, without the assumption that $(\partial p)/(\partial y) = 0$, a new total differential equation is derived and solved which accounts for the coupling between the momentum and energy equations through the body-force terms. The criterion for the importance of this coupling and the conditions for obtaining similar solutions are clearly established. Flow over a constant temperature horizontal flat plate, which does not admit a similar solution, is considered briefly with the Karman-Pohlhausen integral method.

In a two-dimensional flow wherein higher-order terms concerning density and viscosity variations are neglected, the momentum equations are

$$-\frac{\partial p}{\partial x} = \rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] - \left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right] + \rho g_x \quad (1)$$

$$-\frac{\partial p}{\partial y} = \rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] - \left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)\right] + \rho g_y \quad (2)$$

The equation of state is assumed to be

$$\rho = \rho_o\left[1 - \beta\left(t - t_o\right)\right] \quad (3)$$

and the energy equation with constant thermal conductivity is

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{k}{\rho C_p}\left(\frac{\partial^2 t}{\partial y^2}\right) + \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right) + \frac{Q}{\rho C_p} \quad (4)$$

where viscous heating and a constant source term have been included.

If one neglects the leading edge where v is large, an order of magnitude analysis reduces Equation (2) to

$$\frac{\partial p}{\partial y} = -\rho g_y \quad (5)$$

With the x axis directed perpendicular to the gravity force, ρg_x is identically zero. When one takes into account the effect of density variations on the body forces alone, and makes the usual assumption that $\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)$ is small, in view of Equation (3), Equations (2) and (5) become

$$-\frac{1}{\rho_o}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{1}{\rho_o}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \quad (6)$$

and

$$-\frac{1}{\rho_o}\frac{\partial p}{\partial y} = \left[1 - \beta\left(t - t_o\right)\right] g_y \quad (7)$$

HORIZONTAL FULLY DEVELOPED FLOW BETWEEN PARALLEL PLATES

At a large distance from the conduit entrance the flow will be fully developed, and for constant viscosity Equation (6) reduces to

$$\frac{1}{\rho_o}\frac{\partial p}{\partial x} = \nu\frac{d^2 u}{dy^2} \quad (8)$$

Differentiating Equation (5) with respect to x and Equation (8) with respect to y and equating the results due

Owing to a misunderstanding the sequel to this paper, which extends the calculations to include the effect of mass transfer, was published previously in the *A.I.Ch.E. Journal*, 8, 423 (1962).

to the continuity of P, one obtains in dimensionless form

$$\frac{d^3\left(\frac{u}{U_o}\right)}{d\lambda^3} = \frac{\beta d^3 g_y}{\nu U_o N_{Re}} \frac{\partial(t-t_o)}{\partial \xi} \quad (9)$$

Since $(u)/(U_o)$ is a function of λ only as the density is assumed constant, except in the body force term, Equation (9) is valid only if the right-hand side is independent of ξ. This condition is clearly satisfied if the temperature distribution is given by

$$t - t_o = A_1\xi + Y(\lambda) \quad (10)$$

The form of Equation (10) is commonly used as the asymptotic solution to the energy equation for fully developed flow. For the present problem it implies constant, but not necessarily equal, heat fluxes at the wall since the radial temperature gradient is independent of ξ. Clearly Equations (9), (10), and the resulting energy equation

$$\frac{u}{U_o}\frac{\partial(t-t_o)}{\partial\xi} = \frac{\partial^2(t-t_o)}{\partial\lambda^2} + \frac{U_o^2\mu}{k}\left[\frac{d\left(\frac{u}{U_o}\right)}{d\lambda}\right]^2 + \frac{Qd^2}{k} \quad (11)$$

are mutually consistent only for the region at great distances from the hydrodynamic and thermal entrances where both the temperature and velocity fields are fully developed.

Substituting Equation (10) into Equation (9) one obtains

$$\frac{d^3\left(\frac{u}{U_o}\right)}{d\lambda^3} = \frac{\beta d^3 g_y A_1}{\nu U_o} = \frac{N_{Gr}}{N_{Re}} = N_1 \quad (12)$$

Obviously the Grashof number used above is based on the axial temperature gradient $A_1 = \frac{\partial t}{\partial\left(\frac{x}{d}\right)}$, which has the dimensions of temperature.

If one uses the conditions of no slip at the walls together with the continuity condition

$$\int_0^1 \left(\frac{u}{U_o}\right) d\lambda = 1$$

the solution of Equation (12) is

$$\frac{u}{U_o} = 6(\lambda-\lambda^2) + \frac{N_1}{24}(4\lambda^3 - 6\lambda^2 + 2\lambda) \quad (13)$$

The first term of Equation (13) is the Hagen-Poiseuille distribution and is therefore the forced convection contribution.

The bottom plate corresponds to $\lambda = 0$; consequently from Equation (13) the friction factors for the upper and lower walls are

$$f_1 = -\frac{2}{N_{Re}} \left.\frac{d\left(\frac{u}{U_o}\right)}{d\lambda}\right]_{\lambda=1} = \frac{12}{N_{Re}} - \frac{N_1}{6N_{Re}}$$

and

$$f_o = \frac{2}{N_{Re}} \left.\frac{d\left(\frac{u}{U_o}\right)}{d\lambda}\right]_{\lambda=0} = \frac{12}{N_{Re}} + \frac{N_1}{6N_{Re}}$$

Therefore stagnation points or incipient reversed flows occur at

$$N_{Gr} = 72\, N_{Re}$$

and

$$N_{Gr} = -72\, N_{Re}$$

respectively. It is seen that cooling and heating cause a tendency toward instability at the lower and upper walls respectively. Therefore, as shown for vertical flows (*7, 8, 14*), low N_{Re} horizontal bounded flows with heat transfer are less stable than those at higher N_{Re} for a given N_{Gr}. Several velocity distributions are plotted in Figure 1.

When one considers Equation (10), Equation (4) becomes in dimensionless form

$$\frac{d^2\theta_1}{d\lambda^2} = \frac{u}{U_o} - N_{Br}\left[\frac{d\left(\frac{u}{U_o}\right)}{d\lambda}\right]^2 - N_Q \quad (14)$$

where

$$\theta_1 = \frac{Y(\lambda) - Y(0)}{A_1} = \frac{t - t_{w_o}}{A_1}$$

and the boundary conditions are

$$\theta_1(0) = 0,\ \theta_1(1) = \theta_{1w}$$

or alternately

$$\frac{d\theta_1}{d\lambda}(0) = -\frac{q(0)d}{kA_1},$$

$$\frac{d\theta_1}{d\lambda}(1) = -\frac{q(1)d}{kA_1} \quad (14a)$$

If one uses Equations (13) and (14*a*) and integrates Equation (14)

$$A_1 = \frac{d}{k}[q(0) - q(1)]$$

and

$$\theta_1 = \left(\lambda^3 - \frac{1}{2}\lambda^4 - \frac{1}{2}\lambda\right) + \frac{N_1}{720}(6\lambda^5 - 15\lambda^4 + 10\lambda^3 - \lambda) - N_{Br}\,[6(2\lambda^4 - 4\lambda^3 + 3\lambda^2 - \lambda) - \frac{N_1}{30}(18\lambda^5 - 45\lambda^4 + 40\lambda^3 - 15\lambda^2 + 2\lambda) + \frac{N_1^2}{1440}(12\lambda^6 - 36\lambda^5 + 40\lambda^4 - 20\lambda^3 + 5\lambda^2 - \lambda)] + \frac{N_Q}{2}(\lambda - \lambda^2) + \theta_{1w}\lambda \quad (15)$$

where

$$\theta_{1w} = \frac{t_{w_1} - t_{w_0}}{A_1} = \frac{1}{2} + \frac{N_1}{720} - \frac{1}{1 - \frac{q(1)}{q(0)}} - 6N_{Br} - \frac{N_1 N_{Br}}{15} - \frac{N_1^2 N_{Br}}{1440} - \frac{N_Q}{2} \quad (16)$$

This result gives the relationship between the relevant parameters and shows that N_1, and hence the velocity field, depends on the thermal characteristics of the system. Figures 2, 3, and 4 show the effects of N_1, N_{Br}, N_Q, and θ_{1w} on the temperature profiles. It might be mentioned that Figures 1 and 2 show deviations from the symmetrical profiles in the same direction for heating as those observed experimentally by Woolfenden with turbulent flow in tubes (*10*).

An estimate of the effect of variable viscosity can be made without much difficulty if its axial variation is neglected. For fully developed flow, when one considers the variation of μ with respect to λ only, Equation (12) may be written as

$$\frac{d^2}{d\lambda^2}\left(\frac{\mu}{\mu_{w_0}} \frac{d\left(\frac{u}{U_o}\right)}{d\lambda}\right) = N_1 \quad (17)$$

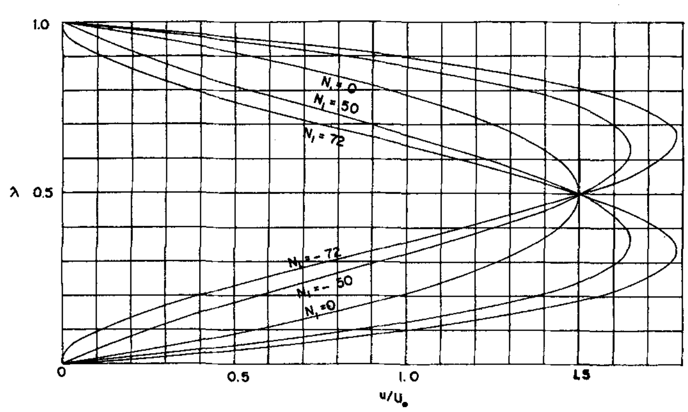

Fig. 1. Velocity distributors for flow between parallel plates for various N_1 with $\epsilon = 0$.

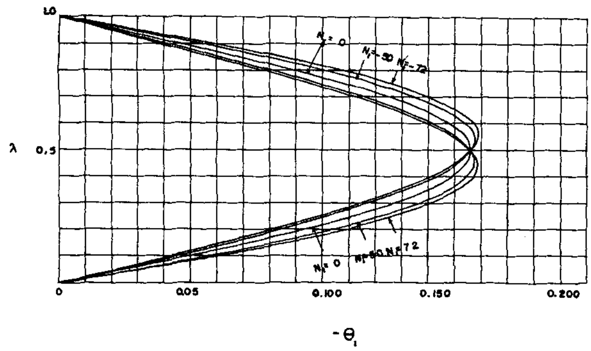

Fig. 2. Temperature distributions for various N_1 with $N_{Br} = N_Q = \theta_{1w} = 0$.

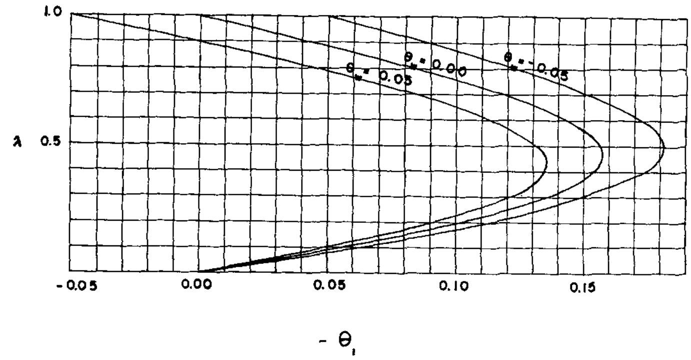

Fig. 4. Temperature profiles for various θ_{1w} with $N_1 = 50$ and $N_{Br} = N_Q = 0$.

where the viscosity used in N_1 is that at the lower wall μ_{w_0}. Let

$$\frac{u}{U_o} = \int_o^\phi \frac{d\phi}{\mu/\mu_{w_0}} \qquad (18)$$

and Equation (14) becomes

$$\frac{d^3\phi}{d\lambda^3} = N_1 \qquad (19)$$

To a good approximation (9) the viscosity of water and liquid sodium, which are particularly important heat transfer media, is given by

$$\frac{\mu_{w_0}}{\mu} = \frac{t + t_a}{t_{w_0} + t_a} = 1 + \frac{t - t_{w_0}}{t_a + t_{w_0}} = 1 + \epsilon\,\theta_1$$

where

$$\epsilon = \frac{A}{t_a + t_{w_0}}$$

Therefore from Equations (15) and (16)

$$\frac{u}{U_o} = \phi + \epsilon \int_o^\lambda \theta_1 \left(\frac{d\phi}{d\lambda}\right) d\lambda \qquad (20)$$

and the energy equation, when one neglects dissipation and source terms for simplicity becomes

$$\frac{d^2\theta_1}{d\lambda^2} = \frac{u}{U_o} = \phi + \epsilon \int_o^\lambda \theta_1 \left(\frac{d\phi}{d\lambda}\right) d\lambda \qquad (21)$$

A method of iteration may now be used to solve Equations (20) and (21) simultaneously. This may be done quite easily. First Equation (19) is integrated, the result together with Equation (15) is substituted into Equation (20), and the arbitrary constants are determined from the boundary conditions. Then Equation (21) may be integrated to obtain a new θ_1 distribution and so on. The calculation has been carried out to a first approximation, and some results are plotted in Figures 5 and 6. It is seen in Figure 5 that the profiles become less stable for cooling and more stable for heating. Figure 6 indicates how the critical value of N_1 is affected by both the magnitude of ϵ and θ_{1w}. It is interesting to note how effective the wall-temperature difference may be in reducing or increasing the value of N_1 required for stagnation, thereby making the flow less or more stable respectively. In general for heating the tendency toward stability is increased through viscosity effects when the upper wall is at a lower temperature than the lower and the reverse is true for cooling.

HORIZONTAL BOUNDARY-LAYER FLOWS

For horizontal boundary-layer flows the assumption of constant physical properties except in the gravitational body force term is retained, and source terms in the energy equation will be neglected. Hence Equation (6) and (7) apply, and Equation (4) reduces to

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 t}{\partial y^2} \qquad (22)$$

The boundary conditions are

$$u(x,0) = v(x,0) = v(x,\infty) = \frac{\partial u(x,\infty)}{\partial y} = 0,\ u(x,\infty) = u_\infty \qquad (23)$$

and

$$t(x,0) = t_w,\quad t(x,\infty) = t_\infty \qquad (24)$$

Similar solutions may be obtained for wedge type of flows where the surface is restricted to a small angle of inclination from the horizontal, since $\rho\, g_x$ is neglected, if the conditions

$$t_w - t_\infty = Bx^n \qquad (25)$$

$$U_\infty = A_3 x^m \qquad (26)$$

where

$$m = \frac{2n+1}{5} \qquad (27)$$

are satisfied. These relations enable one to transform Equations (6), (7), and (22) into two total differential equations by first eliminating the pressure terms in Equations (6) and (7) and then using the dimensionless variables η, f, and θ_2 defined by

$$\eta = \frac{1}{2}\,\frac{y\left(\frac{\bar{u}_\infty}{\nu}\right)}{\left[x\frac{\bar{u}_\infty}{\nu}\right]^{\frac{2-n}{5}}}$$

$$\Psi = \bar{u}_\infty x \left(\frac{x\,\bar{u}_\infty}{\nu}\right)^{\frac{n-2}{5}} f(\eta)$$

and

$$\theta_2 = \frac{t - t_\infty}{t_w - t_\infty}$$

where

$$u_\infty = \bar{u}_\infty \left(\frac{\bar{u}_\infty}{\nu}x\right)^m$$

After performing the necessary trans-

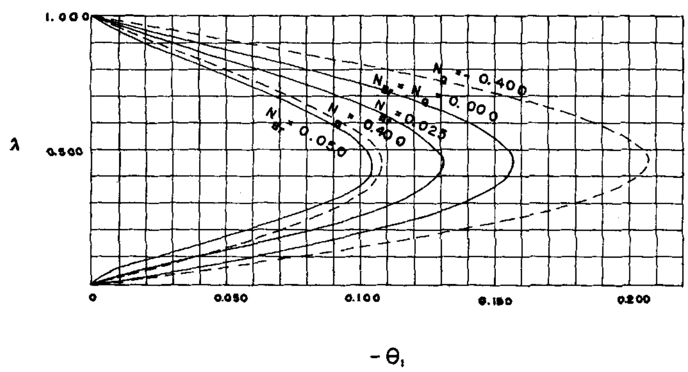

Fig. 3. Effect of N_{Br} and N_Q on temperature distribution for $N_1 = 50$ and $\theta_{1w} = 0$. Solid lines represent $N_Q = 0$ and dashed lines $N_{Br} = 0$.

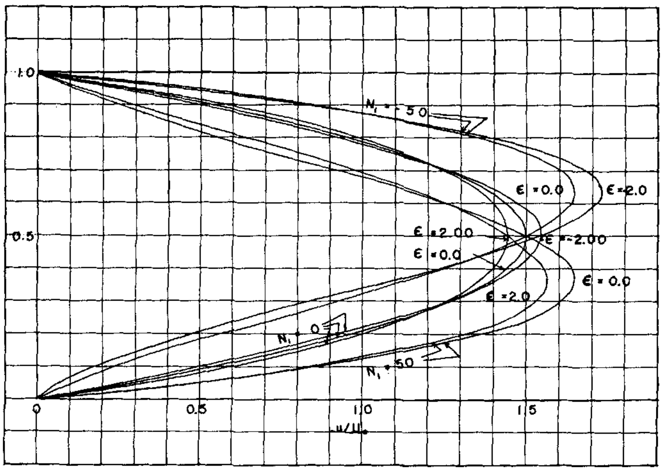

Fig. 5. Effect of viscosity on velocity profiles for various N_1 and ϵ with $N_{Br} = N_Q = \theta_{1w} = 0$.

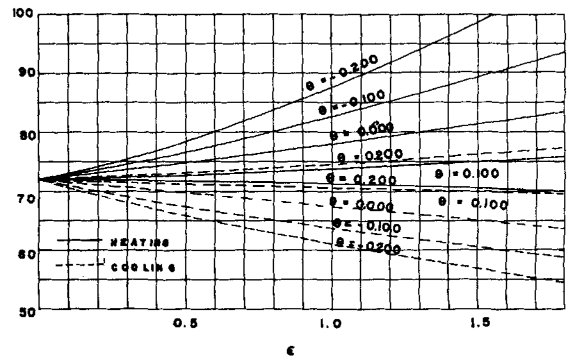

Fig. 6. Effect of viscosity on N_1 required for stagnation for various θ_{1w}.

formation operations on Equations (6), (7), and (22) one has

$$f'''' = \frac{N_2}{2}[(5m-1)\,\theta_2 - (1-m)\,\eta\,\theta_2'] + [(3m-1)\,f'f'' - (m+1)\,ff'''] \quad (28)$$

and

$$\theta_2'' + N_{Pr}[(m+1)\,f\,\theta_2' - (5m-1)\,f'\,\theta_2] = 0 \quad (29)$$

with

$$N_2 = \frac{16\,N_{Gr_x}}{N_{Re_x}^{\,n+3}}\cos\gamma$$

$$N_{Gr_x} = \frac{\beta g_y x^3 (t_w - t_\infty)}{\nu^2}$$

$$N_{Re_x} = \frac{\bar{u}_\infty\, x}{\nu}$$

In contrast to the energy equation obtained by Fage and Falkner (*6*) Equation (29) depends on the potential flow since n and m are not independent. Clearly the boundary conditions are from Equations (23) and (24)

$$f(0) = f'(0) = f''(\infty) = 0; \qquad f'(\infty) = 2 \quad (30)$$

and

$$\theta_2(0) = 1; \; \theta_2(\infty) = 0 \quad (31)$$

Before the authors proceed to the solution of Equation (28), some comments concerning its development and relation to the well-known Falkner-Skan equation seem appropriate since they relate to one of the basic assumptions of the boundary layer theory. Clearly it was necessary to use the boundary condition $\frac{\partial u(x,\infty)}{\partial y} = 0$ which is implied by the boundary-layer theory, because the order of the momentum equations in increased to four if one does not make the usual assumption $(\partial p)/(\partial y) = 0$. However it is a relatively simple matter to relate $f'''(0)$ to m and thereby show rigorously that for isothermal flow over a flat plate or with potential flows in the form of Equation (26) the use of Equation (7) and $\frac{\partial u(x,\infty)}{\partial y} = 0$ leads to the Blausius and Falkner-Skan equations identically. Hence for isothermal flows the relation $(\partial p)/(\partial y) = -\rho g_y$ is implicit in the boundary-layer theory. With the nonisothermal flows over a flat plate considered here it is also necessary to evaluate $f'''(0)$. For finite N_{Pr} it can be shown that $f'''(0) = 0$. It is of course another matter to show the region of applicability of the solutions obtained.

For the horizontal flat plate, on the basis of Carrier and Lin's (*4*) discussion, it is probable that the results obtained from Equation (28) apply except at the leading edge.

The fundamental case of flow past a horizontal flat plate will now be considered with Equation (28). Since u_∞ is constant, $m = 0$. From Equation (27) this yields $n = -\frac{1}{2}$, and Equations (28) and (29) become

$$f'''' + f'f'' + ff''' + N_2\,(\theta_2 + \eta\,\theta_2') = 0 \quad (32)$$

and

$$\theta_2'' + N_{Pr}\,(f\,\theta_2' + f'\,\theta_2) = 0 \quad (33)$$

Although the form of Equation (33) has been considered by several authors (*5*), the solutions do not apply here since it is coupled with Equation (32) which has not previously been derived.

When one integrates, Equation (32) gives

$$f''' + ff'' + N_2\,\eta\,\theta_2 = f'''(0) \quad (34)$$

and Equation (33) becomes

$$\theta_2 = e^{-N_{Pr}\int_0^\eta f d\eta} \quad (35)$$

Equation (35) represents the remarkable case of no heat transfer with a nonuniform temperature field. Eckert (*5*) has discussed similar situations.

By using Equation (30) as previously mentioned, it can be shown that $f'''(0) = 0$, and hence Equations (34) and (35) become the Blausius equation with an added term reflecting the gravitational field effect

$$f''' + ff'' + N_2\,\eta\, e^{-N_{Pr}\int_0^\eta f d\eta} = 0 \quad (36)$$

Numerical solutions to Equation (36) for $N_{Pr} = 0.01, 0.7, 5.0$ have been obtained with a method similar to that suggested by Rheinboldt (*15*). If

$$\Phi = f''$$

then Equation (34) yields after one integration

$$\Phi = \frac{[\Phi(0) - N_2 \int_0^\eta \eta\, e^{(1-N_{Pr})\int_0^\eta f d\eta}\, d\eta]}{e^{\int_0^\eta f d\eta}} \quad (37)$$

$$\Phi(0) = \frac{2 + N_2 \int_0^\infty e^{-\int_0^\eta f d\eta} \left(\int_0^\eta \eta\, e^{(1-N_{Pr})\int_0^\eta f d\eta}\, d\eta\right) d\eta}{\int_0^\infty e^{-\int_0^\eta f d\eta}\, d\eta} \quad (38)$$

By partial integration

$$\int_0^\eta f\, d\eta = \int_0^\eta \frac{(\eta - \sigma)^2}{2}\,\Phi(\sigma)\, d\sigma$$

Hence one may iterate on ϕ using Equations (37) and (38) until $\phi(0)$ is obtained to the desired accuracy, and then an additional integration will yield the velocity distribution.

Some rather interesting numerical results are plotted in Figure 7. Inflections in the velocity profiles occur for

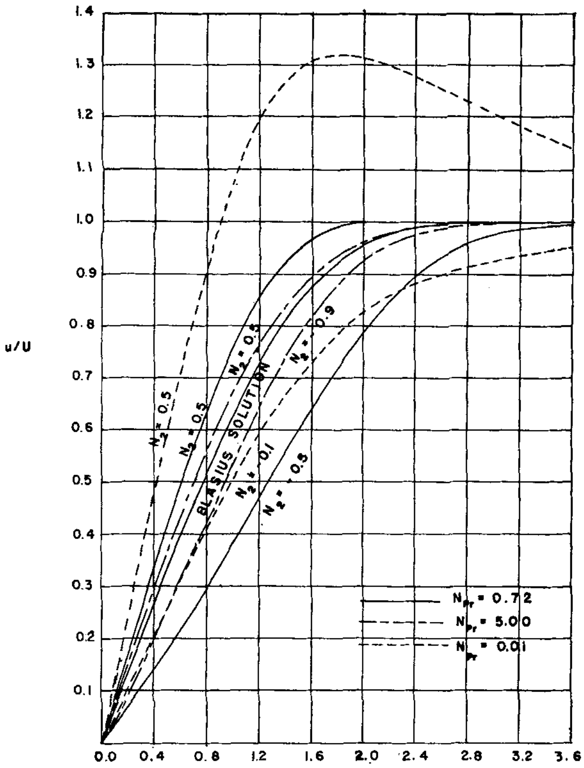

Fig. 7. Boundary layer velocity profiles for various N_2 and N_{Pr}.

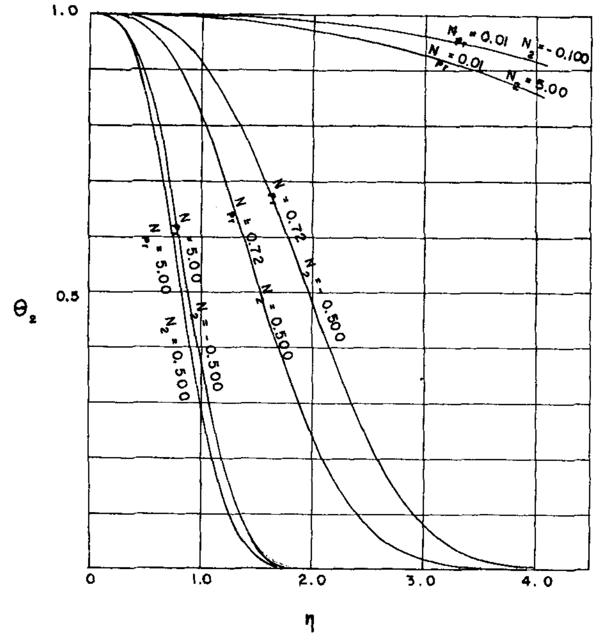

Fig. 8. Boundary layer temperature profiles for various N_2 and N_{Pr}.

negative N_2 and indicate a tendency toward flow instability, whereas positive N_2 tends to stabilize the flow. Clearly buoyancy effects are most pronounced at lower N_{Pr}. For positive N_2 boundary-layer velocities were found to exceed those in the free stream and then smoothly return to the free-stream value. This effect is very obvious for $N = 0.5$ and $N_{Pr} = 0.01$. Brown (2, 3) has reported a similar effect from inertial density variations under rather extreme conditions. Figure 8 gives some corresponding temperature distributions.

It appears that for $N_{Pr} = 1$, $\Phi(0) = 0$, $\left[\frac{\partial u(0)}{\partial y} = 0\right]$ for negative values of N_2 between 0.67 and 0.68, and in this range the value of $\phi(0)$ changes very rapidly with N_2. Similar behavior was observed for the other N_{Pr} studied, and the absolute value of N_2 required for stagnation decreases as N_{Pr} decreases.

It might be mentioned that for the special case of $m = 1$ similar solutions may be obtained with the body force term retained in the x-direction momentum equation also. However the results of this have not been investigated here.

The Constant Temperature Plate

For this case no suitable similarity substitutions were found. Therefore to obtain some approximate information integral methods were used. For convenience different dimensionless coordinates are introduced initially, and Equations (6) and (7) yield

$$-\frac{\partial^2 p_1}{\partial x_1 \partial y_1} = \frac{\partial}{\partial y_1}\left(u_1 \frac{\partial u_1}{\partial x_1}\right) + \frac{\partial}{\partial y_1}\left(v_1 \frac{\partial u_1}{\partial y_1}\right) - \frac{\partial^3 u_1}{\partial y_1^3} = -\frac{\nu \beta g_y}{u_\infty^3}\frac{\partial (t - t_\infty)}{\partial x_1} = -N_3 \frac{\partial \theta_2}{\partial x_1} \quad (39)$$

If one integrates Equation (39)

$$\left[\frac{\partial^2 u_1}{\partial y_1^2}\right]_{y_1=0} = -N_3 \frac{\partial}{\partial x_1}\int_0^\delta \theta_2 \, dy_1 \quad (40)$$

Now the assumed velocity and temperature profiles are taken to be

$$u_1 = \left[\frac{3}{2}\left(\frac{y_1}{\delta}\right) - \frac{1}{2}\left(\frac{y_1}{\delta}\right)^3\right] + \frac{3N_3}{32}\delta^2 \frac{d\delta_T}{dx_1}\left[\frac{y_1}{\delta} - 2\left(\frac{y_1}{\delta}\right)^2 + \left(\frac{y_1}{\delta}\right)^3\right] \quad (41)$$

and

$$\theta_2 = 1 - \frac{3}{2}\left(\frac{y_1}{\delta_T}\right) + \frac{1}{2}\left(\frac{y_1}{\delta_T}\right)^3 \quad (42)$$

Equations (41) and (42) were derived in the usual way except that the velocity distribution was required to satisfy Equation (40).

The pressure gradient distribution for use in the integral momentum equation follows from integrating Equation (39) and using the condition $\left[\frac{\partial P_1}{\partial x_1}\right]_{y_1=0} = \left[\frac{\partial^2 u_1}{\partial y_1^2}\right]_{y_1=0}$ obtained from Equation (6). It may be written by means of Equation (42) as

$$-\frac{\partial P_1}{\partial x_1} = N_3 \frac{d\delta_T}{dx_1}\left[\frac{3}{8} - \frac{3}{4}\left(\frac{y_1}{\delta_T}\right)^2 + \frac{3}{8}\left(\frac{y_1}{\delta_T}\right)^4\right] \quad (43)$$

Equation (43) shows that $(\partial P_1)/(\partial x_1) = 0$ at $y_1 = \delta_T$, and the pressure gradient is zero outside the thermal boundary layer.

The use of Equations (41), (42), and (43) in the integral momentum equation yields

$$\frac{d}{dx_1}\left\{\delta\left[\frac{9N_3^2\delta^4\left(\frac{d\delta_T}{dx_1}\right)^2}{(32^2)\,(105)} + \frac{3N_3\delta^2\left(\frac{d\delta^2}{dx_1}\right)}{32\,(140)} - \frac{39}{280}\right]\right\} - \frac{N_3}{5}\delta_T\frac{d\delta_T}{dx_1} + \frac{3}{2\delta} + \frac{3N_3}{32}\delta\frac{d\delta_T}{dx_1} = 0$$

and the integral energy equation gives

$$\frac{d}{dx_1}\left[\delta\left(\frac{3}{20}\Delta^2 - \frac{3}{280}\Delta^4\right) + \frac{3N_3}{32}\delta^3\frac{d\delta_T}{dx_1}\left(\frac{\Delta^2}{10} - \frac{\Delta^3}{12} + \frac{3}{140}\Delta^4\right)\right] = \frac{1.5}{\delta_T N_{Pr}}$$

These nonlinear equations have not

been solved; however some qualitative information can be obtained. From Equation (41) $\left[\frac{du_1}{dy_1}\right]_{y_1=0} = 0$ when $N_3 = -(48)/(\delta)\,(d\delta_T)/(dx_1)$. Since δ and $(d\delta_T)/(dx)$ are positive, N_3 is negative and again negative N_3 tends to make the flow less stable.

It is quite probable that the integral method can be extended with some success to investigate the entrance flow between horizontal parallel plates. This should be a rather interesting case, since the boundary layers on the upper and lower walls will be influenced differently by the body forces. Also the acceleration of the central core should tend to stabilize the system.

DISCUSSION

Further investigation of the numerical solutions to Equation (28) is presently being carried out. Calculations which include the effect of finite interfacial velocities show that body forces for positive N_2 markedly reduce the blowing velocities required to obtain stagnation at the surface.

Since N_3 is inversely proportional to u_∞^3, it is clear that body forces in horizontal boundary-layer flows are important only for relatively low free-stream velocities. For vertical flows Acrivos (*1*) found them to be inversely proportional to u_∞^2.

Further investigation of the combined effects of inertial density and viscosity variations in conjunction with body force effects may provide additional insight into some important factors relating to stability.

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NOTATION

A_1 = axial temperature gradient defined by Equation (8)
A_2 = $A_1/N_{Pe} = \partial t/\partial\left(\frac{x}{d}\right)$
A_3 = constant defined by Equation (26)
B = constant defined by Equation (25)
c_p = heat capacity
d = distance between plates
f_o = friction factor for lower wall
f_1 = friction factor for upper wall
f = dimensionless stream function
g_x = x component of gravitational force
g_y = y component of gravitational force
k = thermal conductivity
m = constant defined by Equation (23)
N_{Br} = Brinkman number, $(\mu U_o^2)/(kA_1)$
N_{Gr} = Grashof number for flow in parallel plates, $\beta d^3 g_y A_2/\nu^2$
N_{Gr_x} = local Grashof number for boundary layer flow, $g_y x^3(t_w - t_\infty)/\nu^2$
N_{Re} = Reynolds number for flow between parallel plates, $(U_o d)/(\nu)$
N_{Re_x} = local Reynolds number for boundary layer flow, $(\bar{U}_\infty x/(\nu)$
N_{Pe} = $N_{Re} \cdot N_{Pr}$
N_{Pr} = Prandtl number, $(C_p\mu)/(k)$
N_Q = dimensionless heat source parameter, $(Qd^2)/(kA)$
N_1 = $(N_{Gr})/(N_{Re})$
N_2 = $(16N_{Gr_x})/(N_{Re_x}{}^{n+3})$
N_3 = $(N_{Gr_x})/(N_{Re_x})$
n = constant defined by Equation (22)
P = hydrodynamic pressure
P_1 = dimensionless pressure, $(P)/(\rho u_\infty^2)$
Q = heat generated per unit volume
t = temperature
t_a = reference temperature for viscosity
t_o = initial temperature free-stream temperature
t_w = wall temperature
t_{wo} = lower wall temperature for $\lambda = 0$
u = local x component of velocity
u_∞ = free-stream velocity
u = constant with dimensions of velocity
u_1 = dimensionless x component of velocity $\left(\frac{u}{u_\infty}\right)$
U_o = bulk-mean velocity
v = local y component of velocity
v_1 = dimensionless y component of velocity (v/u_∞)
x = distance along the surface
y = distance normal to the surface
x_1 = $(xu_\infty)/(\nu)$
$Y(\lambda)$ = radial dimensionless temperature distribution function defined by Equation (8)
y_1 = $(yu\ )/(\nu)$

Greek Letters

β = coefficient of expansion
γ = angle of attack
δ = dimensionless thickness of momentum boundary layer
δ_T = dimensionless thickness of thermal boundary layer
Δ = δ_T/δ
ϵ = $A_1/(t_a + t_{wo})$
η = dimensionless similarity variable
θ_1 = dimensionless temperature for flow between parallel plates
θ_2 = dimensionless temperature for boundary-layer flow
θ_{1w} = dimensionless wall temperature difference, $\theta_{1w} = \frac{t_w - t_{wo}}{A_1}$
λ = dimensionless radial coordinate (y/d)
μ = viscosity
μ_{wo} = viscosity of lower wall ($\lambda = 0$)
ν = kinematic viscosity
ξ = dimensionless axial coordinate (x/dN_{Pe})
ρ = density
ρ_o = initial density
ρ_∞ = free-stream density
Φ = $(d^2f)/(d\eta^2)$
ϕ = defined by Equation (15)
Ψ = stream function defined by $u = (\partial\Psi)/(\partial y)$, $v = -(\partial\Psi)/(\partial x)$

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